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Statistical mechanics of classical dilute relativistic plasmas in equilibrium

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Abstract. Within a fully relativistic framework, the thermodynamics of a classical dilute arbitrarily hot plasma in equilibrium is studied. The internal energy of the plasma is calculated to all orders in KT/mc^2. The case of a one-component plasma immersed in a static background is also studied. Up to order kT/mc^2 the results given previously by the authors are recovered. On the other hand we also give explicit expressions for the thermodynamic functions of a high-temperature electron-positron plasma. Some important questions concerning the coherence of our calculations and those of other authors are discussed.

1. Introduction

Hitherto, the statistical study of classical dilute relativistic plasmas (CDRP) has only been done for not too high temperatures. The reason is clearly that it has been widely believed for a long time that the Darwin approximation (which is only valid for not too high velocities) can not be improved upon without exact reference to the degrees of freedom of the electromagnetic field. The consequence of this has been that satisfactory classical relativistic statistical mechanics (CRSM) for such systems has not existed until recently.

We now have a formulation of CRSM constructed by Lapiedra and Santos (1981) which, without any independent field degrees, goes beyond any low-velocity approximation. The above authors used predictive relativistic mechanics (PRM, Bel and Martín 1975) in order to describe the microphysics of a system of N relativistic interacting particles. PRM postulates Newton-like equations of motion for the particles, in such a way that the degrees of freedom of the interaction fields are taken into account in the accelerations but do not appear explicitly. Hence a CRSM resembling the non-relativistic analogy has been built into the above reference by Lapiedra and Santos.

However the accelerations appearing in PRM are not exactly known. In fact, for the electromagnetic interaction, they can be written as a series expansion in powers of the dimensionless parameter $\varepsilon_h = e^2/mh$ (Lapiedra *et al* 1979), e, m being the typical charge and mass of the particles and h the mean impact parameter (we take the speed of light $c = 1$). Then in order to ensure fast convergence of these expansions we need relatively high impact parameters. In a plasma, h is of the same order as the mean distance between particles $\rho^{-1/3}$ (ρ being the density), and so the plasmas we deal with will be dilute enough. In fact for actual dilute plasmas only the first term in ε_h

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of these expansions is needed. Notice that since no expansions of the velocities are involved, the CRSM given by Lapiedra and Santos (1981) can be used to describe arbitrarily hot plasmas so long as they are dilute. Therefore we can study not only 'Darwin' plasmas but any hot plasma where the creation-annihilation of pairs can be neglected. This could be the case of a dilute enough electron-positron plasma, which could be considered to be momentarily in equilibrium without taking into account the annihilation of pairs and the resulting photons.

In this paper we are concerned with a two-component CDRP in equilibrium. This problem has been dealt with in the slightly relativistic domain by some authors, mainly Krizan and Havas (1962), Krizan (1974), Kosachev and Trubnikov (1969), Trubnikov and Kosachev (1968), Trubnikov (1968), Lapiedra and Santos (1983) and Barcons and Lapiedra (1983). Krizan and Havas started with the Darwin interaction which at first sight seems able to describe the thermodynamics of a plasma up to order kT/m (remember that the Darwin approximation is valid up to order v^2 and that in a plasma $v^2 \sim kT/m$). Then these authors calculated the two-particle distribution function and the energy of the plasma by summing the ring diagrams. As pointed out in Barcons and Lapiedra (1983), the ring summation need not necessarily be correct in the relativistic case, because the presence of long-range correlations does not ensure its convergence. On the other hand, if one chooses the standard decoupling of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy as the method for the determination of the two-particle distribution function within the Darwin approximation, one is led to a non-screened correlation (see § 4). As a consequence the internal energy per particle in the plasma becomes divergent in the thermodynamic limit. Therefore one could think that the thermodynamical properties of a plasma up to order kT/m cannot be obtained from the Darwin approximation. Our exact (i.e. valid for all temperatures) calculation in this paper, will clarify this point. We conclude (see §§ 4 and 6) that in order to obtain the thermodynamic functions of a plasma up to order kT/m, we must start with a microscopic interaction beyond the v^2-approximation. Therefore the results of Krizan and Havas are not correct.

On the other hand, in the above papers, Trubnikov and Kosachev start their calculations with the Darwin Lagrangian, but when going to the corresponding Hamiltonian they retain some terms beyond order v^2. These terms are of the same order as others which come from the terms neglected in doing the approximation of the Darwin Lagrangian. In other words, their method would only be justified if the Darwin Lagrangian was an exact description of the classical electromagnetic particle interaction, which is obviously not the case. Hence their basic interaction and results are not correct.

Finally, in Lapiedra and Santos (1983), from the standard decoupling of the exactly relativistic BBGKY hierarchy in a dilute plasma, a two-particle distribution function has been given. Then Barcons and Lapiedra (1983), by using this distribution function and retaining only terms up to order kT/m, obtained a correction to the internal energy of the plasma. In the present paper we give an exact expression for the energy which reduces to that previously obtained by Barcons and Lapiedra up to order kT/m. So we are led to the conclusion that these previous results are the correct ones (see §§ 4 and 6).

The plan of the paper is as follows. In § 2, we start with a brief review of the CRSM given in Lapiedra and Santos (1981) and then we calculate the internal energy of a two-component CDRP valid for all temperatures. In § 3 we discuss the problem of the calculation of the equation of state, which is not analytically possible for all temperatures.

In § 4 we examine carefully the slightly relativistic limit where comparison with the above mentioned works is done. Section 5 is devoted to the study of the high-temperature limit and its application to electron–positron plasmas. In § 6 we discuss our results and some important questions about the coherence of several approaches to the study of CDRP.

The case of a one-component CDRP of electrons with a mechanically rigid positive background is dealt with in appendix 2. This is a typical case in which equilibrium is not present, but as we shall see, it can be treated as if the neutralising background was absent.

2. The energy of a two-component plasma to all orders in kT/m

Let us consider the $6N$-dimensional space $\boldsymbol{x}_a$, $\boldsymbol{u}_a$, $a = 1, \ldots, N$, $\boldsymbol{x}_a$ being the three-position of particle a, and $\boldsymbol{u}_a$ the spatial components of its four-velocity in a given frame (i.e., $\boldsymbol{u}_a = \gamma_a \boldsymbol{v}_a$ where $\boldsymbol{n}_a$ is the velocity and $\gamma_a = (1 - v_a^2)^{-1/2}$). Let $F^{(N)}(t, \boldsymbol{x}_a, \boldsymbol{u}_a)$ be the distribution function in this space. As is well known in PRM (Fustero and Verdaguer 1981 see also Currie *et al* 1963) the coordinates $\boldsymbol{x}_a$, $\boldsymbol{u}_a$ cannot be canonical for the case of a relativistic interaction. As a consequence $F^{(N)}(t, \boldsymbol{x}_a, \boldsymbol{u}_a)$ will not generally fulfil a Liouville equation, rather its conservation law will be a continuity equation. However it can be seen that in the case of a plasma, $F^{(N)}$ does fulfil a Liouville equation to lowest order in ε_h (Lapiedra and Santos 1981).

Furthermore to this order, the interactions in PRM are binary, since three-particle terms are at least of order ε_h^2 (see Fustero and Verdaguer 1981). So we are concerned with a problem which can be studied with techniques very similar to those used in non-relativistic statistical mechanics. For example, defining the S-particle reduced distribution functions by integration of $F^{(N)}$ over $\Pi_{R=S+1}^{N}\, \mathrm{d}^3x_R\, \mathrm{d}^3u_R$ one finds (Lapiedra and Santos 1981) a hierarchy of equations (the relativistic BBGKY hierarchy).

Now, if the standard Bogoliubov ansatz is assumed

$$\begin{aligned} F^{(2)}(a, b) &= F^{(1)}(a) F^{(1)}(b)[1 + G(a, b)] \\ F^{(3)}(a, b, c) &= F^{(1)}(a) F^{(1)}(b) F^{(1)}(c)[1 + G(a, b) + G(a, c) + G(b, c)] \end{aligned} \tag{1}$$

the hierarchy is cut off. In fact, the approximation (1) corresponds (Barcons and Lapiedra 1983, Van Kampen 1968) to the first term in an expansion of the distribution functions in powers of the dimensionless parameter $\varepsilon_d = c^2\rho^{1/3}/kT$ (i.e., three-body correlations will be of order ε_d^2). So in a dilute plasma we have only two-particle correlations and two-particle interactions.

Now let us restrict ourselves to the case of a homogeneous and isotropic plasma in equilibrium (no external fields are present). We assume also that the plasma is a two-component neutral one, with $N/2$ particles with mass m_1 and charge e, and $N/2$ particles with mass m_2 and charge $-e$. Then for the one-particle distribution function we have

$$F^{(1)}(a) = [\beta' m_a / 4\pi V K_2(\beta' m_a)]\, \mathrm{e}^{-\beta' m_a \gamma_a} \tag{2}$$

where V is the volume of the system, K_2 the second-order modified Bessel function and β' is a parameter which coincides with $\beta = 1/kT$ in the non-relativistic limit or in the free-gas case ($\rho \to 0$ or $T \to \infty$). The reason for the introduction of β' is two-fold. On the one hand, the change of the canonical coordinates to $\boldsymbol{x}_a$, $\boldsymbol{u}_a$ can alter β to yield another parameter β' as explained in Lapiedra and Santos (1981). On the other hand, the fact of having a velocity-dependent interaction can give rise to a similar effect (Kosachev and Trubnikov 1969). It will be shown in § 3 that $\beta'/\beta = 1 + O(\varepsilon_d^{3/2})$.

By using the distribution function (2), Lapiedra and Santos (1983) have obtained the correlation function $G(a, b)$ from the standard decoupling of the BBGKY hierarchy. Its Fourier transform reads as

$$\hat{G}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = \hat{G}_{\mathrm{DH}}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) + \hat{G}_{\mathrm{R}}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{3}$$

where

$$\hat{G}_{\mathrm{DH}}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = -(2/\pi)^{1/2} \frac{\beta' e_1 e_2}{K^2 + \mathscr{H}^2} \tag{4a}$$

$$\varkappa^2 = 4\pi\beta'\rho e^2 \tag{4b}$$

and

$$\hat{G}_{\mathrm{R}}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\beta' e_1 e_2}{K^2 + \alpha^2} \times \frac{[1 + (\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2)][(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2) - \boldsymbol{v}_1\cdot\boldsymbol{v}_2]}{[1 - (\boldsymbol{n}\cdot\boldsymbol{v}_1)^2][1 - (\boldsymbol{n}\cdot\boldsymbol{v}_2)^2]}, \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{K}}{K} \tag{5a}$$

$$\alpha^2 = \tfrac{1}{2}\varkappa^2 \left(\frac{K_0(\beta' m_1)}{(\beta' m_1)^2 K_2(\beta' m_1)} + \frac{K_0(\beta' m_2)}{(\beta' m_2)^2 K_1(\beta' m_2)} \right) \tag{5b}$$

with K_0 and K_2 the modified zeroth and second-order Bessel functions. Clearly G_{DH} corresponds to the Debye–Huckel correlation function except for the change $\beta \to \beta'$, and G_{R} is a relativistic correction.

On the other hand, for the microscopic energy of two charges we have

$$H(1,2) = H^{(0)}(1) + H^{(0)}(2) + H^{(1)}(1,2) + O(\varepsilon_h^2) \tag{6}$$

where

$$H^{(0)}(a) = m_a \gamma_a \tag{7}$$

and the Fourier transform of $H^{(1)}(1,2)$ is (see appendix 1)

$$\hat{H}^{(1)}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = (2/\pi)^{1/2} \frac{e_1 e_2}{K^2} \left(\frac{1}{1 - (\boldsymbol{n}\cdot\boldsymbol{v}_1)^2} + \frac{1}{1 - (\boldsymbol{n}\cdot\boldsymbol{v}_2)^2} - \frac{[1 - \boldsymbol{v}_1\cdot\boldsymbol{v}_2][1 + (\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2)]}{[1 - (\boldsymbol{n}\cdot\boldsymbol{v}_1)^2][1 - (\boldsymbol{n}\cdot\boldsymbol{v}_2)^2]} \right). \tag{8}$$

Now, we want to calculate the internal energy of the plasma, i.e.

$$E = \int F^{(N)}(t, \boldsymbol{x}_1, \boldsymbol{u}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{u}_N) H(\boldsymbol{x}_1, \boldsymbol{u}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{u}_N) \prod_{a=1}^{N} \mathrm{d}^3 x_a\, \mathrm{d}^3 u_a. \tag{9}$$

In the above reference the present authors have given some plausible arguments to show that up to order $\varepsilon_d^{3/2}$, this energy has only four terms:

$$E=\sum_a \int H^0(a)F^{(1)}(a)\,\mathrm{d}^3x_a\,\mathrm{d}^3u_a \tag{10a}$$

$$+\sum_{a,b}\int [H^0(a)+H^0(b)]F^{(1)}(a)F^{(1)}(b)G(a,b)\,\mathrm{d}^3x_a\,\mathrm{d}^3u_a\,\mathrm{d}^3x_b\,\mathrm{d}^3u_b \tag{10b}$$

$$+\sum_{a,b}\int H^{(1)}(a,b)F^{(1)}(a)F^{(1)}(b)\,\mathrm{d}^3x_a\,\mathrm{d}^3u_a\,\mathrm{d}^3x_b\,\mathrm{d}^3u_b \tag{10c}$$

$$+\sum_{a,b}\int H^{(1)}(a,b)F^{(1)}(a)F^{(1)}(b)G(a,b)\,\mathrm{d}^3x_a\,\mathrm{d}^3u_a\,\mathrm{d}^3x_b\,\mathrm{d}^3u_b \tag{10d}$$

where (10a) is an ε_d^0 term (except for the change $\beta\to\beta'$), (10b) and (10c) are order ε_d and (10d) is order $\varepsilon_d^{3/2}$. (For a detailed explanation see Barcons and Lapiedra (1983) where E, given by (10a)-(10d) has been calculated up to order kT/m. Here we give the exact calculation in kT/m.)

Now let us proceed with the calculation of these terms. The integral (10a) is the energy of an ideal gas except for the change $\beta\to\beta'$. Then the corresponding energy per particle is, in evident notation

$$\frac{E_a}{N}\equiv\frac{1}{\beta'}-\frac{1}{4}m_1\frac{K_3(\beta' m_1)+K_1(\beta' m_1)}{K_2(\beta' m_1)}-\frac{1}{4}m_2\frac{K_3(\beta' m_2)+K_1(\beta' m_2)}{K_2(\beta' m_2)}. \tag{11}$$

For a neutral plasma (as in our case), the second and third terms (10b) and (10c) vanish. For a non-neutral plasma they would give a divergent contribution to the energy per particle. In other words, the thermodynamic limit does not exist for a non-neutral plasma.

Now, the term (10d) because of the homogeneity of the plasma (i.e. $H^{(1)}(a,b)$ and $G(a,b)$ only depend on $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$ through its difference $\boldsymbol{x}_a-\boldsymbol{x}_b$) can be written as

$$E_d\equiv V\sum_{a,b}\int \mathrm{d}^3u_a\,\mathrm{d}^3u_b\,\mathrm{d}^3K\,F^{(1)}(a)F^{(1)}(b)\hat{G}(\boldsymbol{K},\boldsymbol{v}_a,\boldsymbol{v}_b)\hat{H}^{(1)}(\boldsymbol{K},\boldsymbol{v}_a,\boldsymbol{v}_b) \tag{12}$$

where we have used the fact that $\hat{H}^{(1)}$ and $\hat{G}$ are even functions of $\boldsymbol{K}$. If we go to spherical coordinates in d^3K, the integration over K can be easily performed and it is convergent. This is so, because of the natural renormalisation provided by the correlation function. Then the contribution due to the first term $H^{(1)}G_{\mathrm{DH}}$ can be evaluated without too many difficulties to give

$$E_{d\mathrm{DH}}/N\equiv-\sqrt{\pi}e^3\rho^{1/2}\beta'^{1/2} \tag{13}$$

which, except for the change $\beta\to\beta'$, is the correction to the energy for a non-relativistic Coulomb plasma (cf Landau and Lifshitz 1967). The calculation of the remaining term (i.e. the integral containing $\hat{H}^{(1)}\hat{G}_{\mathrm{R}}$) is very tediuos and must be done going to polar coordinates in the variables $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, choosing $\boldsymbol{n}$ as the z axis. The result is

$$E_{d\mathrm{R}}/N\equiv-\frac{\pi\beta'\rho e^4}{\alpha}\Bigg[\bigg(\frac{1}{\beta' m_1}\frac{K_1(\beta' m_1)}{K_2(\beta' m_1)}+\frac{K_0(\beta' m_1)}{(\beta' m_1)^2K_2(\beta' m_1)}+\frac{1}{\beta' m_2}\frac{K_1(\beta' m_2)}{K_2(\beta' m_2)}+\frac{K_0(\beta' m_2)}{(\beta' m_2)^2K_2(\beta' m_2)}\bigg)^2+\bigg(\frac{K_0(\beta' m_1)}{(\beta' m_1)^2K_2(\beta' m_1)}+\frac{K_0(\beta' m_2)}{(\beta' m_2)K_2(\beta' m_2)}\bigg)^2\Bigg] \tag{14}$$

where the notation should be evident.

Hence, the sum of equations (12), (13) and (14) gives the energy per particle in a CDRP for all temperatures. However this expression is purely formal as long as we do not know what β' is. This problem is discussed in the next section.

In the case of a one-component plasma (with a static neutralising background) the results are essentially the same, as we shall see in appendix 2.

3. The virial theorem and the equation of state

As we shall see in this section, the pressure, as well as the parameter β' cannot be analytically evaluated for all temperatures. Sections 4 and 5 are devoted to the study of this problem in the 'Darwin' and ultrarelativistic limits. However for the sake of coherence of our theory we present here a method to evaluate the pressure and β' in the general case.

It is obvious that β' must become β when the density of the plasma vanishes. Therefore in a dilute plasma $\beta'/\beta = 1 + O(\varepsilon_d^{\eta})$ where η is some positive number (we shall see that $\eta = \frac{3}{2}$). Hence if in the calculation of the energy only $\varepsilon_d^{3/2}$ terms have been kept, to the same approximation we can substitute $\beta' = \beta$ in equations (13) and (14). Then if we define $C \equiv \beta' - \beta$, for the energy per particle we have in evident notation

$$\frac{E}{N} \simeq \frac{1}{\beta}\left(1 - \frac{C}{\beta}\right) - \frac{1}{4}\left[\!\!\left[\left\{ m_1 \frac{K_3(\beta m_1) + K_1(\beta m_1)}{K_2(\beta m_1)} + \frac{1}{2} C m_1 \right.\right.\right.$$
$$\left. \times\left[\frac{K_4(\beta m_1) + 2K_2(\beta m_1) + K_0(\beta m_1)}{K_2(\beta m_1)} - \left(\frac{K_3(\beta m_1) + K_1(\beta m_1)}{K_2(\beta m_1)}\right)^2\right]\right\}$$
$$\left. + (1 \leftrightarrow 2)\right]\!\!\right] + (13) + (14) \tag{15}$$

where in (13) and (14) β' can be substituted by β and where only linear terms in C have been retained. This is so because, as we said above, C will be seen to be of order $\varepsilon_d^{3/2}$.

Now, let us calculate the pressure. From the equation (Barcons and Lapiedra 1983)

$$p = -\frac{1}{\beta}\left(\frac{\partial \beta'}{\partial V}\right)_{\beta} E(\beta') - \frac{1}{\beta}\int^{\beta'} \left(\frac{\partial E(\beta'')}{\partial V}\right)_{\beta''} \mathrm{d}\beta'' + \frac{\rho}{\beta} \tag{16}$$

we have

$$p = \frac{\rho}{\beta} - \frac{N}{\beta}\left(\frac{\partial C}{\partial \beta}\right)_V \left(\frac{1}{\beta} - \frac{1}{4} m_1 \frac{K_3(\beta m_1) + K_1(\beta m_1)}{K_2(\beta m_1)} - \frac{1}{4} m_2 \frac{K_3(\beta m_2) + K_1(\beta m_2)}{K_2(\beta m_2)}\right)$$
$$- \frac{1}{3}\sqrt{\pi}\, e^3 \rho^{3/2} \beta^{1/2} - \frac{1}{\beta} \frac{\pi e^4 \rho^{3/2}}{2} \int^{\beta} \mathrm{d}\beta''\, \beta'' f(p'')/\bar{\alpha}(\beta'') \tag{17}$$

where

$$f(\beta) = \left(\frac{1}{\beta m_1} \frac{K_1(\beta m_1)}{K_2(\beta m_1)} + \frac{1}{(\beta m_1)^2} \frac{K_0(\beta m_1)}{K_2(\beta m_1)} + (1 \leftrightarrow 2)\right)^2 + \left(\frac{K_0(\beta m_1)}{(\beta m_1)^2 K_2(\beta m_1)} + (1 \leftrightarrow 2)\right)^2, \tag{18a}$$

$$\bar{\alpha}(\beta) = \alpha(\beta, \rho)\rho^{-1/2} \tag{18b}$$

are two functions depending on βm_1 and βm_2 but not on ρ (see equation (5b)).

Now in order to know what β' is, we use the Virial Theorem (Lapiedra and Barcons 1983, cf Landau and Lifshitz 1970), valid for systems with electromagnetic interaction without radiation

$$E - 3pV = \sum_{a=1}^{N} m_a\langle(1 - v_a^2)^{1/2}\rangle. \tag{19}$$

Subtracting from (19) a similar equation for the ideal gas case, we can find a differential equation for C which reads:

$$\rho(\partial C/\partial\rho)_\beta A(\beta) = CB(\beta) + \rho^{1/2}D(\beta) \tag{20}$$

where

$$D(\beta) = \frac{-\pi\beta e^4}{\bar{a}(\beta)} f(\beta) + \frac{3\pi e^4}{2\beta}\int^{\beta} \mathrm{d}\beta''\, \frac{\beta'' f(\beta'')}{\bar{a}(\beta'')} \tag{21a}$$

$$A(\beta) = \frac{3}{\beta}\left(\frac{1}{\beta} - \frac{m_1}{4}\,\frac{K_3(\beta m_1) + K_1(\beta m_1)}{K_2(\beta m_1)} - \frac{m_2}{4}\,\frac{K_3(\beta m_2) + K_1(\beta m_1)}{K_2(\beta m_2)}\right) \tag{21b}$$

$$B(\beta) = \frac{1}{\beta^2} + \left[\frac{m_1^2}{8}\left(\frac{4K_2(\beta m_1) + 3K_0(\beta m_1) + K_4(\beta m_1)}{K_2(\beta m_1)} - \frac{K_3^2(\beta m_1) + 3K_1^2(\beta m_1) + 4K_1(\beta m_1)K_3(\beta m_1)}{K_2(\beta m_1)}\right) + (1 \leftrightarrow 2)\right] \tag{21c}$$

are again only β-dependent functions. Then the differential equation (20) must be solved together with the boundary condition $C(\rho = 0) = 0$. Under this condition, it is not difficult to see that the solution is

$$C(\rho) = \{2D(\beta)/[A(\beta) + 2B(\beta)]\}\rho^{1/2} \tag{22}$$

from which we conclude that, as we said before, $C(\rho) \sim O(\varepsilon_d^{3/2})$ for all temperatures. On the other hand, notice that for a purely Coulomb plasma, the term $D(\beta)$ is identically zero, and so the correction $C(\rho)$ vanishes, i.e., β' coincides with β in this case.

Therefore, we see that our calculation is consistent with the Coulomb limit, though it cannot be analytically carried out in the general case. If one were interested in a case not included in those considered in the next sections, numerical calculations should be performed.

4. The Darwin plasma: thermodynamics up to order kT/m

Let us consider in this section a plasma whose dimensionless parameter kT/m is small enough such that $k(kT/m)^2$ terms can be neglected. Then for the internal energy of this plasma we obtain from equations (11), (13) and (14) the following result

$$\frac{E}{N} \simeq M + \frac{3}{2\beta}\left(1 - \frac{C}{\beta}\right) + \frac{15}{16\beta^2\mu} - (\pi\beta\rho)^{1/2}e^3 - \left(\frac{\pi\rho}{2}\right)^{1/2}\frac{e^3}{\mu\beta^{1/2}} \tag{23}$$

where $M = (m_1 + m_2)/2$ and $\mu = m_1 m_2/(m_1 + m_2)$. We have used the asymptotic expansions of the modified Bessel functions given in Abramowitz and Stegun (1965). This result coincides with that previously given by the authors in the above reference. Therefore the approximations done in this reference in order to obtain the thermodynamic functions up to order kT/m are actually correct.

Now, if we proceed as explained in the previous section, we can obtain the following results (Lapiedra and Barcons 1983)

$$\beta' \simeq \beta\left[1-\frac{4}{3}\left(\frac{\pi}{2}\right)^{1/2}\frac{e^3\rho^{1/2}}{(kT)^{3/2}}\frac{(kT)^2}{M\mu}\right] \tag{24}$$

$$p \simeq kT\rho\left[1-\frac{\sqrt{\pi}}{3}\frac{e^3\rho^{1/2}}{(kT)^{3/2}}\left(1+\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right)\right]. \tag{25}$$

For the sake of completeness we also give the expressions found for the specific heat $C_V=(1/N)\partial E/\partial T)_V$ and the compressibility $K_T=-V^{-1}(\partial V/\partial p)_\beta$:

$$C_V \simeq K\left[\frac{3}{2}\left(1+\frac{5}{4}\frac{kT}{\mu}\right)+\frac{\sqrt{\pi}}{2}\frac{e^3\rho^{1/2}}{(kT)^{3/2}}\left(1-\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right)\right] \tag{26}$$

$$K_T^{-1}=\rho kT\left[1-\frac{\sqrt{\pi}}{2}\frac{e^3\rho^{1/2}}{(kT)^{3/2}}\left(1+\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right)\right]. \tag{27}$$

Let us now return to the point raised in the introduction according to which these results cannot be derived from the Darwin approximation. As is well known, the Darwin interaction is valid up to order v^2 and therefore it could be thought that it can describe the thermodynamics of a plasma up to order kT/m. In what follows we try to explain why this is not true.

Let us describe the microscopic interaction in the plasma by the Darwin Lagrangian (cf Landau and Lifshitz 1970) and use the BBGKY hierarchy with the standard decoupling for the distribution function given by equations (1). Then for the correlation function $G(1,2)$, one finds (Lapiedra and Santos 1981) the integer-differential equation

$$\boldsymbol{v}_1\cdot\left(\frac{\partial G(1,2)}{\partial \boldsymbol{x}_1}-\beta' m_1\gamma_1^{-1}\xi_{12}-\beta' m_1\gamma_1^{-1}\sum_{R=3}^{N}\int \xi_{1R}F^{(1)}(R)G(2,R)\,\mathrm{d}^3x_R\,\mathrm{d}^3u_R\right)$$
$$+((1\leftrightarrow 2)=0 \tag{28}$$

where ξ_{ab} are the space components of the four-acceleration, which the charge b produces on charge a, as derived from the Darwin Lagrangian. (See again the above reference for an expression for ξ_{ab}.) Obviously, in equation (28), to the approximation we deal with, γ_a can be taken as $1+\frac{1}{2}v_a^2$.

Fourier transformation of equation (28) reads:

$$\hat{G}(\boldsymbol{K},\boldsymbol{v}_1,\boldsymbol{v}_2)(\boldsymbol{n}\cdot\boldsymbol{v}_1-\boldsymbol{n}\cdot\boldsymbol{v}_2)+\left(\frac{4\pi\beta' e_1}{K^2}\sum_{R=3}^{N}e_R\int \mathrm{d}^3u_R\,\hat{G}(\boldsymbol{K},\boldsymbol{v}_2,\boldsymbol{v}_1)F^{(1)}(R)\right.$$
$$\left.\times\{\boldsymbol{n}\cdot\boldsymbol{v}_1+(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_R)^2-(\boldsymbol{n}\cdot\boldsymbol{v}_R)(\boldsymbol{v}_1\cdot\boldsymbol{v}_R)\}-(1\leftrightarrow 2)\right)$$
$$=-(2/\pi)^{1/2}(\beta' e_1e_2/K^2)[\boldsymbol{n}\cdot\boldsymbol{v}_1+(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2)^2$$
$$-(\boldsymbol{n}\cdot\boldsymbol{v}_2)(\boldsymbol{v}_1\cdot\boldsymbol{v}_2)-(1\leftrightarrow 2)]. \tag{29}$$

This inhomogeneous integral equation allows for the unshielded solution

$$\hat{G}(\boldsymbol{K},\boldsymbol{v}_1,\boldsymbol{v}_2)=-\left(\frac{2}{\pi}\right)^{1/2}\frac{e_1e_2\beta'}{K^2+\varkappa^2}+\left(\frac{2}{\pi}\right)^{1/2}\frac{\beta' e_1e_2}{K^2}[(\boldsymbol{n}\cdot\boldsymbol{v}_1)(\boldsymbol{n}\cdot\boldsymbol{v}_2)-\boldsymbol{v}_1\cdot\boldsymbol{v}_2] \tag{30}$$

whose Fourier transform gives rise to the unphysical correlation function given in Lapiedra and Santos (1981). It can be easily seen that this correlation function gives no correction of order kT/m to the energy of a Coulomb plasma. Furthermore, if using (30) one wants to go to higher orders in kT/m one finds a divergent energy per particle when $N\to\infty$.

Now, it is not difficult to see that the only solution of (29) which is quadratic in the velocities is (30). Therefore if one is searching for a dipole-dipole correlation function (Krizan 1974) one is led necessarily to (30). On the other hand, it seems to be natural that for $k\to\infty$ the influence of the particles labelled $R=3\ldots N$ be negligible, and so for K large enough one must have $\hat{G}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = -\beta'\hat{H}_D(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2)$ where $\hat{H}_D$ is the Fourier transform of the Darwin Hamiltonian. Then if one searches for a solution of (29) in powers of $1/k^2$, $-\beta'\hat{H}_D$ being the first term in this expansion, one finds again equation (30). (Notice that this expansion corresponds formally to a series in powers of e^2/K^2.) Let us remark that equation (30) can also be obtained by a coarse-graining technique (Goldstein 1969).

Hence we arrive at the conclusion that at this level, the only solution to (29) which can be retained, is the correlation function (30). Actually, our exactly relativistic correlation function (3), (4), (5) reduces to (30) when v^4 and $(kT/m)^2$ terms are neglected. However, when these terms are kept, we find a correction to the thermodynamic functions of order kT/m. As a consequence, we can say that the thermodynamics of a Darwin plasma (i.e. a plasma where $(kT/m)^2$ terms cannot be neglected) cannot be worked out using the Darwin interaction, in the sense that higher-order terms in the velocities are needed in the basic microscopic interaction, to get a correction of order kT/m. A more detailed discussion of this important result is carried out in § 6.

5. The ultrarelativistic limit: electron–positron plasma

Let us consider a plasma for which we have $\beta m \ll 1$. At these temperatures the creation of electron-positron pairs occurred and we can consider that our plasma is mainly composed of electrons and positrons in the same proportion. So in order to study the thermodynamics of such a plasma, we can use the expressions given in §§ 2 and 3 for $m_1 = m_2 = m$ and $m\beta \ll 1$. If we take into account the expansions given in Abramowitz and Stegun (1965) for the modified Bessel functions in the limit of small argument, we have

$$E/N \simeq (3/\beta)(1-C/\beta) - (2\pi)^{1/2} e^3 \rho^{1/2} \beta^{1/2} \log^{3/2}(2/\beta m) \tag{31}$$

where only dominant terms in the kinetic, as well as in the interaction energy have been kept. We see from (31) that in the ultrarelativistic limit $\beta m \ll 1$ the Debye-Hückel interaction term (13) does not play any role.

On the other hand, to the same order, performing the standard calculations explained in § 3, we arrive at the following results

$$\beta' \simeq \beta \tag{32}$$

$$p \simeq \rho kT\left(1 - \frac{(2\pi)^{1/2}}{3} \frac{e^3\rho^{1/2}}{(kT)^{3/2}} \log^{3/2}\frac{2kT}{m}\right). \tag{33}$$

On the other hand, for the specific heat and the isothermal compressibility we have

$$C_V \simeq 3K\left(1 + \frac{(2\pi)^{1/2}}{6} \frac{e^3\rho^{1/2}}{(kT)^{3/2}} \log^{3/2}\frac{2kT}{m}\right) \tag{34}$$

$$K_T^{-1} \simeq \rho kT\left[1-\left(\frac{\pi}{2}\right)^{1/2}\frac{e^3\rho^{1/2}}{(kT)^{3/2}}\log^{3/2}\frac{2kT}{m}\right]. \tag{35}$$

From equations (31)-(35) we see that in limit $T\to\infty$, we recover, as expected, the thermodynamics of a classical ultrarelativistic ideal gas (i.e., $E/N=3kT$, $C_V=3k$, $P=\rho kT$, $K_T^{-1}=\rho kT$). This shows that also in the high-temperature limit our calculations lead to consistent results.

Notice that the study of the ultrarelativistic limit, given in this section, has been possible because in our framework no expansions in the velocities are involved.

6. Conclusions

We have studied in this paper some features of the thermodynamics of a classical arbitrarily hot plasma in equilibrium in an exactly relativistic framework. In connection with this, there is the point of the covariance of our results, which has not been studied here. The question is the following: as our starting point is exactly Lorentz covariant, how do our results change under a Lorentz transformation? The answer was given by Van Kampen (1969), who showed that distribution functions in phase-space are Lorentz invariant. Actually, in this paper, we worked out our calculations in a concrete advantageous frame of reference in which the plasma is in equilibrium, homogeneous and isotropic. If one were interested in looking at the plasma from another frame of reference, Van Kampen's results would have to be taken into account. For example it can be seen that the energy of a plasma behaves like the temporal component of a four-vector whose components corresponding to the three-momentum are zero in our frame. Then the extension of our results is straightforward. Nevertheless, in this paper, we are not concerned with this problem.

Now let us come back to the problem of the Darwin plasma studied in § 4. As we said there, it is not the same to approximate the microscopic model and then to work out the thermodynamics, than to work out first the thermodynamics and then to approximate the final results. Clearly when, as in this case, the two ways do not coincide, it seems to be more reasonable to follow the second way. Now, in Barcons and Lapiedra (1983) the following heuristic point of view has been advanced: when statistical calculations must be performed and one must start with a given approximated microscopic interaction up to a certain order, one must proceed systematically to perform the statistic without neglecting or introducing terms of the same order than those considered in the microscopic interaction. The hope is then that, in this way, one will obtain the approximated thermodynamics of the system up to a given order, which corresponds to the order kept in the starting microscopic interaction, i.e., it is assumed that higher-order terms neglected in the microscopic interaction, if they were kept, would give rise to higher orders in the macroscopic description. Nevertheless, this heuristic point of view is not correct for a Darwin plasma, since, according to which has been explained in § 4, in this case, beside v^2 terms, v^4 terms need also to be considered at least in the basic interaction in order to get a correction of order kT/m in the thermodynamic functions. Which is then the real situation? As we said in § 4, when starting from the Darwin interaction, one is led to an unphysical, infrared divergent correlation function, which gives an infinite energy per particle, i.e., the thermodynamic limit does not exist in this case. Thus, for equilibrium systems driven by long-range interactions, it can happen that, starting from the approximated microscopic interaction to a given order and doing the appropriate statistical calculations one does not find the expected thermodynamic limit. Then one must probably consider higher-order terms in the microscopic description in order to ensure the existence of this limit. This is just what we have done for a slightly relativistic plasma in the above reference.

Dealing with this problem, Trubnikov and Kosachev have kept all terms in v which come from the Darwin Lagrangian when one tries to obtain the velocities as functions of the positions and generalised momenta. In this way, they forget all terms which come from the corrections to the Darwin Lagrangian itself. These terms are of the same order as those which have been retained by then. Thus their approximation is not consistent.

Finally, let us remark that the good behaviour of our results in the low- and high-temperature limits, convinces us that the method developed here and in the papers referred to by Barcons, Lapiedra and Santos is the correct one to deal with CDRP in equilibrium.

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Appendix 1

In this appendix we derive equation (8) in the framework of PRM. The expression for $H^{(1)}$ $(\boldsymbol{x}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2)$ has been derived in Bel and Martin (1975) but it is too complicated for our purposes. Therefore, we will obtain the Fourier transform of $H^{(1)}(\boldsymbol{x}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2)$ without using the explicit expression of $H^{(1)}$.

Consider in the PRM formalism the following system of differential equations governing the evolution of two interacting particles $a = 1, 2$

$$\mathrm{d}\boldsymbol{x}_a/\mathrm{d}t = \boldsymbol{v}_a, \qquad \mathrm{d}\boldsymbol{v}_a/\mathrm{d}t = \boldsymbol{a}_a(\boldsymbol{x}_b, \boldsymbol{v}_c). \tag{A1.1}$$

Poincaré invariance of this manifestly predictive system is ensured only if the accelerations $\boldsymbol{a}_a$ satisfy the Currie–Hill conditions (Currie 1966, Hill 1967). On the other hand the condition which ensures translational invariance reads:

$$\sum_{a=1}^{2} \frac{\partial a_b^i}{\partial x_a^\kappa} = 0 \tag{A1.2}$$

which says that $\boldsymbol{a}_a$ can only depend on $\boldsymbol{x}_1, \boldsymbol{x}_2$ through $\boldsymbol{x}_{12} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.

Now the Lie algebra associated with the Poincaré group, can be represented as a set of ten generating functions H, $\boldsymbol{P}$, $\boldsymbol{J}$, $\boldsymbol{K}$, which have the usual meaning of total energy, total momentum, total angular momentum and centre of mass position. Concretely it can be seen that H fulfils the following equation (Fustero and Verdaguer 1981)

$$H = H^{(0)} + \int_0^{-\infty} \mathrm{d}\lambda \; R(\lambda) \alpha_b^i \frac{\partial H}{\partial v_b^i} \tag{A1.3}$$

(summation over b is understood), where

$$H^{(0)} = \sum_{a=1}^{2} m_a \gamma_a \tag{A1.4}$$

and $R(\lambda)$ is an operator acting as follows

$$R(\lambda) f(\boldsymbol{x}_a, \boldsymbol{v}_b) = f(\boldsymbol{x}_a + \lambda \boldsymbol{v}_a, \boldsymbol{v}_b). \tag{A1.5}$$

It can be seen in PRM formalism that from the Poincaré invariance we have

$$\sum_{a=1}^{2} \frac{\partial H}{\partial x_a^K} = 0 \tag{A1.6}$$

i.e., H depends only on $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ through $\boldsymbol{x}_{12}$.

Now let us assume that the two particles interact via electromagnetism and that a perturbative expansion for the accelerations in powers of ε_h makes sense

$$a_a^i = \sum_{n=1}^{\infty} a_a^{i(n)} \tag{A1.7}$$

where $a_a^{i(n)}$ is of order ε_h^n. This leads us to a similar expansion for H:

$$H = \sum_{n=0}^{\infty} H^{(n)}. \tag{A1.8}$$

We are now interested in the evaluation of $H^{(1)}$. From (A1.3) we have

$$H^{(1)} = \frac{\partial H^{(0)}}{\partial v_b^i} \int_0^{-\infty} \mathrm{d}\lambda\, R(\lambda) a_b^{i(1)}. \tag{A1.9}$$

For $a_a^{i(1)}$ we have the following expression (Bel *et al* 1973)

$$\begin{aligned} a_a^{i(1)}(\boldsymbol{x}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2) &= \frac{e_a e_{a'}}{m_a} \gamma_a^{-1} (-1)^{a+1} [(1 - \boldsymbol{v}_a \cdot \boldsymbol{v}_{a'}) \delta_j^i + v_{a'}^i v_{aj} - v_a^i v_{aj}] \\ &\quad \times \frac{\gamma_{a'} x_{12}^i}{[\boldsymbol{x}_{12}^2 + (\boldsymbol{x}_{12} \cdot \boldsymbol{u}_{a'})^2]^{3/2}}, \qquad a' \neq a \end{aligned} \tag{A1.10}$$

whose Fourier transform reads:

$$\begin{aligned} \hat{a}_a^{i(1)}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) &= \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}x_{12} \exp(\mathrm{i}\boldsymbol{K}, \boldsymbol{x}_{12}) a_a^{i(1)}(\boldsymbol{x}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2) \\ &= i \left(\frac{2}{\pi}\right)^{1/2} \frac{e_a e_{a'}}{m_a} (-1)^{a+1} \\ &\quad \times \frac{(1 - \boldsymbol{v}_a \cdot \boldsymbol{v}_{a'}) K^i + (\boldsymbol{K} \cdot \boldsymbol{v}_a - \boldsymbol{K} \cdot \boldsymbol{v}_{a'}) v_{a'}^i + [(\boldsymbol{K} \cdot \boldsymbol{v}_{a'})(\boldsymbol{v}_a \cdot \boldsymbol{v}_{a'}) - \boldsymbol{K} \cdot \boldsymbol{v}_a]}{K^2 - (\boldsymbol{K} \cdot \boldsymbol{v}_{a'})^2}. \end{aligned} \tag{A1.11}$$

Now taking Fourier transforms in equation (A9) we find

$$\hat{H}^{(1)}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) = \sum_{a=1}^{2} m_a \gamma_a^3 \left(\frac{1}{\mathrm{i}\boldsymbol{K} \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)} - \pi \delta(\boldsymbol{K} \cdot \boldsymbol{v}_1 - \boldsymbol{K} \cdot \boldsymbol{v}_2) \right) \boldsymbol{v}_a \cdot \hat{\boldsymbol{a}}_a^{(1)}(\boldsymbol{K}, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{A1.12}$$

from which equation (8) can be easily derived.

Appendix 2

In this appendix we shall study the thermodynamics of a plasma with N electrons at an arbitrarily relativistic temperature T and N positive neutralising ions at rest (usually protons). This case has been extensively dealt with in the literature (Krizan and Havas 1962, Krizan 1974, Trubnikov and Kosachev 1968) but its correct justification has not been done to our knowledge, at least in the relativistic case.

First let us consider distribution functions involving protons. In an obvious notation we have

$$F_{\mathrm{p}}^{(1)}((a)=(1/V)\delta^{(3)}(\boldsymbol{u}_a) \tag{A2.1}$$

because protons are always at rest. On the other hand it is not difficult to see that the joint distribution functions $F_{\mathrm{pp}}^{(2)}(a,b)$ and $F_{\mathrm{pe}}^{(2)}(a,b)$ fulfil

$$F_{\mathrm{pp}}^{(2)}(a,b)=F_{\mathrm{p}}^{(1)}(a)F_{\mathrm{p}}^{(1)}(b) \tag{A2.2}$$

$$F_{\mathrm{pe}}^{(2)}(a,b)=F_{\mathrm{p}}^{(1)}(a)F_{\mathrm{e}}^{(1)}(b). \tag{A2.3}$$

Equation (A2.2) is evident because the protons do not have any motion. In order to justify equation (A2.3) consider the identity

$$F_{\mathrm{pe}}^{(2)}(a,b)=W_{\mathrm{pe}}(a/b)F_{\mathrm{e}}^{(1)}(b) \tag{A2.4}$$

where $W_{\mathrm{pe}}(a/b)$ is the conditional probability density that the ion be at $\boldsymbol{x}_a$, $\boldsymbol{u}_a$ in phase space when the electron is at $\boldsymbol{x}_b$, $\boldsymbol{u}_b$. Clearly as the ion is always at the same place, $W_{\mathrm{pe}}(a/b)$ does not depend on the position and velocity of the electron, and so $W_{\mathrm{pe}}(a/b)=F_{\mathrm{p}}^{(1)}(a)$.

Now let us consider the expression of the total energy of the plasma written in the following way

$$E=\sum_a\int \mathrm{d}^6a\, H^{(0)}(a)F^{(1)}(a) \tag{A2.5}$$

$$+\sum_{a,b}\int \mathrm{d}^6a\,\mathrm{d}^6b\, H^{(1)}(a,b)F^{(2)}(a,b) \tag{A2.6}$$

where the sums must be extended, in principle, to electrons and ions. However we shall see that except for a constant we can restrict these sums only for electrons.

First the term (A2.5) gives

$$Nm_{\mathrm{p}}+N\left(\frac{1}{\beta^1}-\frac{1}{2}\frac{K_3(\beta'm)+K_1(\beta'm)}{K_2(\beta'm)}\right) \tag{A2.7}$$

where m_{p} is the mass of one of the ions and m the electron mass.

Let us put $H^{(1)}(a,b)=H_{\mathrm{c}}(a,b)+H_{\mathrm{R}}(a,b)$, where H_{C} is the purely Coulombic part and H_{R} is the relativistic part which goes to zero when one of the two velocities involved $\boldsymbol{u}_a$, $\boldsymbol{u}_b$ vanishes. Therefore (A2.6) becomes

$$(\mathrm{A2.6})=\sum_{a,b}\int \mathrm{d}^6a\,\mathrm{d}^6b\, H_{\mathrm{c}}(a,b)F^{(2)}(a,b) \tag{A2.8}$$

$$+\sum_{\substack{a,b\\ \text{electrons}}}\int \mathrm{d}^6a\,\mathrm{d}^6b\, H_{\mathrm{R}}(a,b)F_{\mathrm{ee}}^{(2)}(a,b). \tag{A2.9}$$

We split again term (A2.8) into four parts, according to (A2.2) and (A2.3)

$$(A2.8) = \sum_{\substack{a,b \\ \text{electrons}}} \int d^6a\, d^6b\, H_c(a,b) F_e^{(1)}(a) F_e^{(1)}(b) G_{ee}(a,b)$$

$$+ \sum_{\substack{a,b \\ \text{electrons}}} \int d^6a\, d^6b\, H_c(a,b) F_e^{(1)}(a) F_e^{(1)}(b)$$

$$+ \sum_{\substack{a,b \\ \text{protons}}} \int d^6a\, d^6b\, H_c(a,b) F_p^{(1)}(a) F_p^{(1)}(b)$$

$$+ \sum_{\substack{a \text{ electrons} \\ b \text{ protons}}} \int d^6a\, d^6b\, H_c(a,b) F_e^{(1)}(a) F_p^{(1)}(b). \tag{A2.10}$$

It is not difficult to see that because of the neutrality of the plasma, the sum of the three last terms of equations (A2.10) vanishes. On the other hand it can be also seen that

$$\int d^6a\, d^6b\, H_R(a,b) F_e^{(1)}(a) F_e^{(1)}(b) = 0 \tag{A2.11}$$

in which case

$$(A2.6) = \sum_{\substack{a,b \\ \text{electrons}}} \int d^6a\, d^6b\, H^{(1)}(a,b) F_e^{(1)}(a) F_e^{(1)}(b) G_{ee}(a,b). \tag{A2.12}$$

Finally, in order to see what $G_{ee}(a, b)$ is, one can remember that $G_{ep}(a, b) = G_{pp}(a, b) = 0$ in which case $G_{ee}(a, b)$ can be calculated as if protons were absent. Therefore the energy per electron can be calculated to give

$$E/N = mp + \frac{1}{\beta'} - \frac{1}{2}\frac{K_3(\beta' m) + K_1(\beta' m)}{K_2(\beta' m)} - \sqrt{\pi} e^3 \rho^{1/2} \beta^{1/2}$$
$$- \frac{4\pi\beta\rho e^4}{\alpha}\left[\left(\frac{1}{\beta m}\frac{K_1(\beta m)}{K_2(\beta m)} + \frac{K_0(\beta m)}{(\beta m)^2 K_2(\beta m)}\right)^2 + \left(\frac{K_0(\beta m)}{(\beta m)^2 K_2(\beta m)}\right)^2\right] \tag{A2.13}$$

$$\alpha^2 = 4\pi\beta\rho e^2[K_0(\beta m)/(\beta m)^2 K_2(\beta m)] \tag{A2.14}$$

i.e., the background only contributes to the energy per electron with its rest mass, which is the expected result.

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